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Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces

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Abstract

In this paper we introduce a broad class of nonlinear mappings which contains the class of contractive mappings and the class of generalized hybrid mappings in a Hilbert space. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Their results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].

1 Introduction

Let H be a real Hilbert space. A mapping T from H into H is said to be contractive if there exists a real number α with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for any $x, y \in H$. By Banach [2] it is known that any contraction mapping has a unique fixed point. Let C be a non-empty subset of H . A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$. By Baillon [1] we know the following nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a nonexpansive mapping from C into C with a fixed point. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point of T .

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping T from C into H is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$$

for any $x, y \in C$; see Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping can be deduced from an equilibrium problem in a Hilbert space; see Blum and Oettli [3] and Combettes and Hirstoaga [5]. Recently Kohsaka and Takahashi [12], and Takahashi [16] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping T from C into H is said to be nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. A mapping T from C into H is said to be hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. Motivated by these mappings, Kocourek, Takahashi and Yao [10] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β) -generalized hybrid mapping. We observe that the class of the mappings above covers the classes of well-known mappings. For example, an (α, β) -generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [11] and Iemoto and Takahashi [7]. Moreover Kocourek, Takahashi and Yao [10] proved the following nonlinear ergodic theorem.

Theorem 1.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H , let T be a generalized hybrid mapping from C into C which has a fixed point, and let P be the metric projection of H onto the set of fixed points of T . Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T , where $p = \lim_{n \rightarrow \infty} PT^n x$.

In this paper we introduce a broad class of nonlinear mappings T from C into H which contains the class of contractive mappings and the class of generalized hybrid mappings. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. These results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H . We denote by $\overline{\text{co}}A$ the closure of the convex hull of A . In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \quad (1)$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [15]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \quad (2)$$

for any $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H . We denote by $F(T)$ the set of fixed points of T . A mapping T from C into H with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [8]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \rightarrow z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \rightarrow 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then, we have from (1) that

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\ &= 0. \end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C . It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in C$; see [15] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [17].

Lemma 2.1. *Let D be a nonempty closed convex subset of H . Let P be the metric projection from H onto D . Let $\{u_n\}$ be a sequence in H . If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ . If μ is a Banach limit on l^∞ , then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [14] for the proof of existence of a Banach limit and its other elementary properties.

Using means and the Riesz theorem, we can obtain the following result; see [13] and [14].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞ . Then there exists a unique point $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H.$$

3 Fixed point theorems

Let H be a real Hilbert space and let C be a nonempty subset of H . A mapping T from C into H is said to be widely generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \\ + \max\{\varepsilon \|x - Tx\|^2, \zeta \|y - Ty\|^2\} \leq 0 \end{aligned} \quad (3)$$

for any $x, y \in C$; see [9]. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [10] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = 0$. We first prove a fixed point theorem for widely generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into itself which satisfies the following conditions (1) and (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$;
- (2) $\varepsilon + \alpha + \gamma > 0$, or $\zeta + \alpha + \beta > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Remark 3.2. We can also prove Theorem 3.1 by using the following condition instead of the condition (2):

$$(2)' \quad \varepsilon - \beta - \delta > 0, \text{ or } \zeta - \gamma - \delta > 0.$$

In the case of the condition $\varepsilon - \beta - \delta > 0$, we obtain from (1) that

$$\varepsilon - \beta - \delta \leq \varepsilon + \alpha + \gamma.$$

Thus we obtain the desired result by Theorem 3.1. Similarly, for the case of $\zeta - \gamma - \delta > 0$, we can obtain the result by using the case of $\zeta + \alpha + \beta > 0$.

As a direct consequence of Theorem 3.1, we obtain the following.

Theorem 3.3. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into itself which satisfies the following conditions (1) and (2):

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0;$$

$$(2) \quad \varepsilon + \alpha + \gamma > 0, \text{ or } \zeta + \alpha + \beta > 0.$$

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Note that an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping T above with $\alpha = 1$, $\beta = \gamma = \varepsilon = \zeta = 0$ and $-1 < \delta < 0$ is a contractive mapping. Using Theorem 3.1, we can show the Banach fixed point theorem in a Hilbert space.

Theorem 3.4 (the Banach fixed point theorem). Let H be a real Hilbert space and let T be a contractive mapping from H into H , that is, there exists a real number α with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for any $x, y \in H$. Then T has a unique fixed point.

Using Theorem 3.1, we can show the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.5 (Kocourek, Takahashi and Yao [10]). Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a generalized hybrid mapping from C into C , that is, there exist real numbers α and β such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for any $x, y \in C$. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded.

Example 3.6. Let H be the real line and let T be a mapping from H into H defined by $Tx = 2x$ for any $x \in H$. Taking $\alpha = 1$, $\beta = \gamma = -2$, $\delta = 4$ and $\varepsilon = \zeta = 2$, we have that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ & \quad + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \\ & = |2x - 2y|^2 - 2|x - 2y|^2 - 2|2x - y|^2 + 4|x - y|^2 + \max\{2x^2, 2y^2\} \\ & = 8|x - y|^2 - 2|(x - y) - y|^2 - 2|x + (x - y)|^2 + \max\{2x^2, 2y^2\} \\ & = -2x^2 - 2y^2 + \max\{2x^2, 2y^2\} \leq 0 \end{aligned}$$

for any $x, y \in H$. Furthermore, since $\{T^n 0 \mid n = 0, 1, \dots\} = \{0\}$, (1) $\alpha + \beta + \gamma + \delta = 1 > 0$ and (2) $\varepsilon + \alpha + \gamma = 1 > 0$, we have from Theorem 3.1 that T has a unique fixed point. However T is not a contractive mapping. Moreover, taking $x = 0$ and $y = 1$, we have that for any real numbers α and β ,

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 - \beta\|Tx - y\|^2 - (1 - \beta)\|x - y\|^2 \\ & = 4\alpha + 4(1 - \alpha) - \beta - (1 - \beta) = 3 > 0. \end{aligned}$$

Thus T is not generalized hybrid.

4 Nonlinear ergodic theorem

In this section, using the technique developed by Takahashi [13], we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Before proving the result, we need the following lemmas.

Lemma 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into C which has a fixed point and satisfies the condition:*

$$(2) \quad \varepsilon + \alpha + \gamma > 0, \text{ or } \zeta + \alpha + \beta > 0.$$

Then the set of fixed points of T is closed.

Lemma 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into C which has a fixed point and satisfies the conditions (1) and (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0;$$

$$(2) \quad \varepsilon + \alpha + \gamma > 0, \text{ or } \zeta + \alpha + \beta > 0.$$

Then the set of fixed points of T is convex.

Lemma 4.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into C which has a fixed point and satisfies the conditions (1) and (3):*

$$(1) \alpha + \beta + \gamma + \delta \geq 0;$$

$$(3) \alpha + \gamma > 0, \text{ or } \alpha + \beta > 0.$$

Then T is quasi-nonexpansive.

Theorem 4.4. Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into C which has a fixed point and satisfies the conditions (1) and (2):

$$(1) \alpha + \beta + \gamma + \delta \geq 0;$$

$$(2) \varepsilon + \alpha + \gamma > 0, \text{ or } \zeta + \alpha + \beta > 0;$$

$$(3) \alpha + \gamma > 0, \text{ or } \alpha + \beta > 0;$$

respectively. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} PT^n x$.

Using Theorem 4.4, we can show the following nonlinear ergodic theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.5 (Kocourek, Takahashi and Yao [10]). Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a generalized hybrid mapping from C into C , that is, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for any $x, y \in C$. Suppose that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T , where $p = \lim_{n \rightarrow \infty} PT^n x$ and P is the metric projection from H onto $F(T)$.

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